# 7 The Mean Value Theorem

The mean value theorem is, like the intermediate value and extreme value theorems, an *existence* theorem. It asserts the existence of a point in an interval where a function has a particular behavior, but it does not tell you how to find the point.

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*Theorem 1 Mean Value Theorem. Suppose that the function $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then there is a point $x_0$ in the open interval $(a, b)$ at which $f'(x_0) = [f(b) - f(a)]/(b - a)$.*

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In physical terms, the mean value theorem says that the average velocity of a moving object during an interval of time is equal to the instantaneous velocity at some moment in the interval. Geometrically, the theorem says that a secant line drawn through two points on a smooth graph is parallel to the tangent line at some intermediate point on the curve. There may be more than one such point, as in Fig. 7-1. Consideration of these physical and geometric interpretations will make the theorem believable.

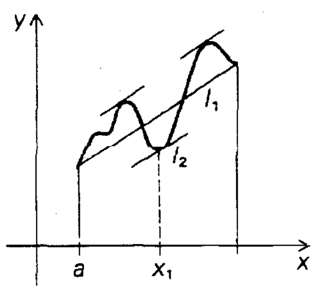


**Fig. 7-1** Tangent lines parallel to the secant line.

We will prove the mean value theorem at the end of this section. For now, we will concentrate on some applications. Our first corollary tells us that if we know something about $f'(x)$ for all $x$ in $[a, b]$, then we can conclude something about the relation between values of $f(x)$ at different points in $[a, b]$.

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*Corollary 1 Let $f$ be differentiable on $(a, b)$ [and continuous on $[a, b]$]. Suppose that, for all $x$ in the open interval $(a, b)$, the derivative $f'(x)$ belongs to a certain set $S$ of real numbers. Then, for any two distinct points $x_1$ and $x_2$ in $(a, b)$ [in $[a, b]$], the difference quotient*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

*belongs to $S$ as well.*

***Proof*** The difference quotient stays the same if we exchange $x_1$ and $x_2$, so we may assume that $x_1 < x_2$. The interval $[x_1, x_2]$ is contained in $(a, b)$ [in $[a, b]$]. Since $f$ is differentiable on $(a, b)$ [continuous on $[a, b]$], it is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. By the mean value theorem, applied to $f$ on $[x_1, x_2]$, there is a number $x_0$ in $(x_1, x_2)$ such that $[f(x_2) - f(x_1)]/(x_2 - x_1) = f'(x_0)$. But $(x_1, x_2)$ is contained in $(a, b)$, so $x_0 \in (a, b)$. By hypothesis, $f'(x_0)$ must belong to $S$; hence so does $[f(x_2) - f(x_1)]/(x_2 - x_1)$.

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***Worked Example 1*** Suppose that $f$ is differentiable on the whole real line and that $f'(x)$ is constant. Use Corollary 1 to prove that $f$ is linear.

***Solution*** Let $m$ be the constant value of $f'$ and let $S$ be the set whose only member is $m$. For any $x$, we may apply Corollary 1 with $x_1 = 0$ and $x_2 = x$ to conclude that $[f(x) - f(0)]/(x - 0)$ belongs to $S$; that is, $[f(x) - f(0)]/(x - 0) = m$. But then $f(x) = mx + f(0)$ for all $x$, so $f$ is linear.

***Worked Example 2*** Let $f$ be continuous on $[1,3]$ and differentiable on $(1,3)$. Suppose that, for all $x$ in $(1,3)$, $1 \leqslant f'(x) \leqslant 2$. Prove that $2 \leqslant f(3) - f(1) \leqslant 4$.

***Solution*** Apply Corollary 1, with $S$ equal to the interval $[1,2]$. Then $1 \leqslant [f(3) - f(1)]/(3 - 1) \leqslant 2$, and so $2 \leqslant f(3) - f(1) \leqslant 4$.

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***Corollary 2*** *Suppose that $f'(x) = 0$ for all $x$ in some open interval $(a, b)$. Then $f$ is constant on $(a, b)$.*

***Proof*** Let $x_1 < x_2$ be any two points in $(a, b)$. Corollary 1 applies with $S$ the set consisting only of the number zero. Thus we have the equation $[f(x_2) - f(x_1)]/(x_2 - x_1) = 0$, or $f(x_2) = f(x_1)$, so $f$ takes the same value at all points of $(a, b)$.

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***Worked Example 3*** Let $f(x) = d|x|/dx$

(a) Find $f'(x)$.

(b) What does Corollary 2 tell you about $f$? What does it not tell you?

*Solution*

(a) Since $|x|$ is linear on $(-\infty, 0)$ and $(0, \infty)$, its second derivative $d^2|x|/dx^2 = f'(x)$ is identically zero for all $x \neq 0$.

(b) By Corollary 2, $f$ is constant *on any open interval on which it is differentiable*. It follows that $f$ is constant on $(-\infty, 0)$ and $(0, \infty)$. The corollary does *not* say that $f$ is constant on $(-\infty, \infty)$. In fact, $f(-2) = -1$, while $f(2) = +1$.

Finally, we can derive from Corollary 2 the fact that two antiderivatives of a function differ by a constant. (An antiderivative of $f$ is a function whose derivative is $f$.)

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***Corollary 3*** *Let $F(x)$ and $G(x)$ be functions such that $F'(x) = G'(x)$ for all $x$ in an open interval $(a, b)$. Then there is a constant $C$ such that $F(x) = G(x) + C$ for all $x$ in $(a, b)$.*

***Proof*** We apply Corollary 2 to the difference $F(x) - G(x)$. Since $(d/dx)[F(x) - G(x)] = F'(x) - G'(x) = 0$ for all $x$ in $(a, b)$, $F(x) - G(x)$ is equal to a constant $C$, and so $F(x) = G(x) + C$.

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***Worked Example 4*** Suppose that $F'(x) = x$ for all $x$ and that $F(3) = 2$. What is $F(x)$?

***Solution*** Let $G(x) = \frac{1}{2}x^2$. Then $G'(x) = x = F'(x)$, so $F(x) = G(x) + C = \frac{1}{2}x^2 + C$. To evaluate $C$, we set $x = 3$: $2 = F(3) = \frac{1}{2}(3^2) + C = \frac{9}{2} + C$. Thus $C = 2 - \frac{9}{2} = -\frac{5}{2}$ and $F(x) = \frac{1}{2}x^2 - \frac{5}{2}$.

## Solved Exercises*

1. Let $f(x) = x^3$ on the interval $[-2, 3]$. Find explicitly the value(s) of $x_0$ whose existence is guaranteed by the mean value theorem.
2. If, in Corollary 1, the set $S$ is taken to be the interval $(0, \infty)$, the result is a theorem which has already been proved. What theorem is it?
3. The velocity of a train is kept between 40 and 50 kilometers per hour

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*Solutions appear in the Appendix.

during a trip of 200 kilometers. What can you say about the duration of the trip?

4. Suppose that $F'(x) = -(1/x^2)$ for all $x \neq 0$. Is $F(x) = (1/x) + C$, where $C$ is a constant?

### Exercises

1. Directly verify the validity of the mean value theorem for $f(x) = x^2 - x + 1$ on $[-1, 2]$ by finding the point(s) $x_0$. Sketch.
2. Suppose that $f$ is continuous on $[0, \frac{1}{2}]$ and $0.3 \leqslant f'(x) < 1$ for $0 < x < \frac{1}{2}$. Prove that $0.15 \leqslant [f(\frac{1}{2}) - f(0)] < 0.5$.
3. Suppose that $f'(x) = x^2$ and $f(1) = 0$. What is $f(x)$?
4. Suppose that an object lies at $x = 4$ when $t = 0$ and that the velocity $dx/dt$ is 35 with a possible error of $\pm 1$, for all $t$ in $[0, 2]$. What can you say about the object's position when $t = 2$?

## Proof of the Mean Value Theorem

Our proof of the mean value theorem will use two results already proved which we recall here:

1. If $x_0$ lies in the open interval $(a, b)$ and is a maximum or minimum point for a function $f$ on an interval $[a, b]$ and if $f$ is differentiable at $x_0$, then $f'(x_0) = 0$. This follows immediately from Theorem 3, p. 64, since if $f'(x_0)$ were not zero, $f$ would be increasing or decreasing at $x_0$.
2. If $f$ is continuous on a closed interval $[a, b]$, then $f$ has a maximum and a minimum point in $[a, b]$ (extreme value theorem, p. 71).

The proof of the mean-value theorem proceeds in three steps.

*Step 1* **(Rolle's Theorem).** *Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$, and assume that $f(a) = f(b) = 0$. Then there is a point $x_0$ in $(a, b)$ at which $f'(x_0) = 0$.*

***Proof*** If $f(x) = 0$ for all $x$ in $[a, b]$, we can choose any $x_0$ in $(a, b)$. So assume that $f$ is not everywhere zero. By result 2 above, $f$ has a maximum point $x_1$ and a minimum point $x_2$. Since $f$ is zero at the ends of the interval but is not identically zero, at least one of $x_1, x_2$ lies in $(a, b)$. Let $x_0$ be this point. By result 1, $f'(x_0) = 0$.

Rolle's theorem has a simple geometric interpretation (see Fig. 7-2).

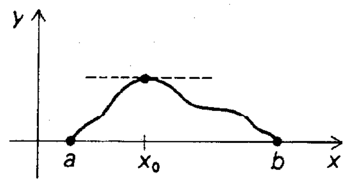


**Fig. 7-2** The tangent line is horizontal at $x_0$.

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*Step 2* **(Horserace Theorem).** *Suppose that $f_1$ and $f_2$ are continuous on $[a, b]$ and differentiable on $(a, b)$, and assume that $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$. Then there exists a point $x_0$ in $(a, b)$ such that $f_1'(x_0) = f_2'(x_0)$.*

***Proof*** Let $f(x) = f_1(x) - f_2(x)$. Since $f_1$ and $f_2$ are differentiable on $(a, b)$ and continuous on $[a, b]$, so is $f$ (see Problem 6 in Chapter 5). By assumption, $f(a) = f(b) = 0$, so from step 1, $f'(x_0) = 0$ for some $x_0$ in $(a, b)$. Thus $f_1'(x_0) = f_2'(x_0)$ as required.

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We call this the horserace theorem because it has the following interpretation. Suppose that two horses run a race starting together and ending in a tie. Then, at some time during the race, they must have had the same velocity.

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*Step 3* We apply step 2 to a given function $f$ and the linear function $l$ that matches $f$ at its endpoints, namely.,

$$l(x) = f(a) + (x - a)\left[\frac{f(b) - f(a)}{b - a}\right]$$

Note that $l(a) = f(a)$, $l(b) = f(b)$, and $l'(x) = [f(b) - f(a)]/(b - a)$. By step 2, $f'(x_0) = l'(x_0) = [f(b) - f(a)]/(b - a)$ for some point $x_0$ in $(a, b)$.

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Thus we have proved the mean value theorem:

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***Mean Value Theorem*** *If $f$ is continuous on $[a, b]$ and is differentiable on $(a, b)$, then there is a point $x_0$ in $(a, b)$ at which*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

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## Solved Exercises

5. Let $f(x) = x^4 - 9x^3 + 26x^2 - 24x$. Note that $f(0) = 0$ and $f(2) = 0$. Show without calculating that $4x^3 - 27x^2 + 52x - 24$ has a root somewhere strictly between 0 and 2.

6. Suppose that $f$ is a differentiable function such that $f(0) = 0$ and $f(1) = 1$. Show that $f'(x_0) = 2x_0$ for some $x_0$ in $(0, 1)$.

## Exercises

5. Suppose that the horses in the horserace theorem cross the finish line with equal velocities. Must they have had the same acceleration at some time during the race?

6. Let $f(x) = |x| - 1$. Then $f(-1) = f(1) = 0$, but $f'(x)$ is never equal to zero on $[-1, 1]$. Does this contradict Rolle's theorem? Explain.

## *Problems for Chapter 7*

1. Suppose that $(d^2/dx^2)\ [f(x) - 2g(x)] = 0$. What can you say about the relationship between $f$ and $g$?

2. Suppose that $f$ and $g$ are continuous on $[a, b]$ and that $f'$ and $g'$ are continuous on $(a, b)$. Assume that

   $f(a) = g(a)$ and $f(b) = g(b)$

   Prove that there is a number $c$ in $(a, b)$ such that the line tangent to the graph of $f$ at $(c, f(c))$ is parallel to the line tangent to the graph of $g$ at $(c, g(c))$.

3. Let $f(x) = x^7 - x^5 - x^4 + 2x + 1$. Prove that the graph of $f$ has slope 2 somewhere between $-1$ and 1.

4. Find the antiderivatives of each of the following:
   (a) $f(x) = \frac{1}{2}x - 4x^2 + 21$
   (b) $f(x) = 6x^5 - 12x^3 + 15x - 11$
   (c) $f(x) = x^4 + 7x^3 + x^2 + x + 1$
   (d) $f(x) = (1/x^2) + 2x$
   (e) $f(x) = 2x(x^2 + 7)^{100}$

5. Find an antiderivative $F(x)$ for the given function $f(x)$ satisfying the given condition:

   (a) $f(x) = 2x^4$; $F(1) = 2$

   (b) $f(x) = 4 - x$; $F(2) = 1$

   (c) $f(x) = x^4 + x^3 + x^2$; $F(1) = 1$

   (d) $f(x) = 1/x^5$; $F(1) = 3$

6. If $f''(x) = 0$ on $(a, b)$, what can you say about $f$?

7. (a) Let $f(x) = x^5 + 8x^4 - 5x^2 + 15$. Prove that somewhere between $-1$ and $0$ the tangent line to the graph of $f$ has slope $-2$.

   (b) Let $f(x) = 5x^4 + 9x^3 - 11x^2 + 10$. Prove that the graph of $f$ has slope 9 somewhere between $-1$ and $1$.

8. Let $f$ be a polynomial. Suppose that $f$ has a double root at $a$ and at $b$. Show that $f'(x)$ has at least three roots in $[a, b]$.

9. Let $f$ be twice differentiable on $(a, b)$ and suppose $f$ vanishes at three distinct points in $(a, b)$. Prove that there is a point $x_0$ in $(a, b)$ at which $f''(x_0) = 0$.

10. Use the mean value theorem to prove Theorem 5, Chapter 5 without the hypothesis that $f'$ be continuous.